

# QUANTUM SYSTEMS IN WEAK GRAVITATIONAL FIELDS

G. PAPINI

*Department of Physics, University of Regina*

*Regina, Sask. S4S 0A2, Canada*

*International Institute for Advanced Scientific Studies*

*Vietri sul Mare (SA), Italy*

AND

*Canadian Institute for Theoretical Astrophysics*

*University of Toronto, On., Canada*

## 1. Introduction

Fully covariant wave equations predict the existence of a class of inertial-gravitational effects that can be tested experimentally. In these equations inertia and gravity appear as external classical fields, but, by conforming to general relativity, provide very valuable information on how Einstein's views carry through in the world of the quantum. Experiments already confirm that inertia and Newtonian gravity affect quantum particles in ways that are fully consistent with general relativity down to distances of  $\sim 10^{-4}cm$  for superconducting electrons [1] and of  $\sim 10^{-8}cm$  for neutrons [2, 3, 4]. Other aspects of the interaction of gravity with quantum systems are just beginning to be investigated.

Gravitational-inertial fields in the laboratory are weak and remain so in the cosmos for most astrophysical sources. These are the fields considered here. They are adequately described by the weak field approximation (WFA).

Gravitational-inertial fields affect particle wave functions in a variety of ways. They induce quantum phases that afford a unified treatment of interferometry and gyroscopy. They interact with particle spins giving rise to a number of significant effects. They finally shift energy levels in particle spectra [5]. While it still is difficult to predict when direct measurements will become possible in the latter case, rapid experimental advances in particle interferometry [6, 7, 8] require that quantum phases be derived

with precision. This will be done below for Schroedinger, Klein-Gordon, Maxwell and Dirac equations. Large, sensitive interferometers hold great promise in many of these investigations. They can play a role in testing general relativity.

Spin-inertia and spin-gravity interactions are the subject of numerous theoretical [9, 10, 11, 12, 13, 14, 15, 16] and experimental efforts [17, 18, 19, 20, 21]. At the same time precise Earth-bound and near space experimental tests of fundamental theories require that inertial effects be identified with great accuracy. It is shown below that spin-rotation coupling is particularly important in precise tests of fundamental theories and in certain types of neutrino oscillations. Surprisingly, particle accelerators may be also called to play a role in these investigations [22].

## 2. Wave equations

The quantum phases induced by inertia and gravity are derived in this section for Schroedinger, Klein-Gordon, Maxwell and Dirac equations. Some applications are given in Section 3.

### 2.1. THE SCHROEDINGER EQUATION

Starting from the action principle ( $\hbar = c = 1$ )

$$S = -m \int ds = -m \int \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} dx^0, \quad (1)$$

where  $\dot{x}^\mu = dx^\mu/dx^0$ , one arrives at the Lagrangian

$$L = -m(g_{ij} \dot{x}^i \dot{x}^j + 2g_{i0} \dot{x}^i + g_{00})^{1/2}, \quad (2)$$

where  $i, j = 1, 2, 3$ . From  $L$  one obtains

$$p_i = \partial L / \partial \dot{x}^i = -m(g_{ij} \dot{x}^j + g_{i0})(g_{lk} \dot{x}^l \dot{x}^k + 2g_{k0} \dot{x}^k + g_{00})^{-1/2}. \quad (3)$$

Substituting (3) into  $H = p_i \dot{x}^i - L$  one finds

$$H = m(g_{i0} \dot{x}^i + g_{i0})(g_{lk} \dot{x}^l \dot{x}^k + 2g_{k0} \dot{x}^k + g_{00})^{-1/2}. \quad (4)$$

In the WFA  $g_{\mu\nu} \simeq \eta_{\mu\nu} + \gamma_{\mu\nu}$ ,  $g^{\mu\nu} \simeq \eta^{\mu\nu} - \gamma^{\mu\nu}$ ,  $g^{ij} g_{jk} \simeq \delta_k^i$ . From (3) one then gets

$$g^{ij} p_j \simeq -m(\dot{x}^i + g^{ij} g_{j0})(g_{lk} \dot{x}^l \dot{x}^k + 2g_{k0} \dot{x}^k + g_{00})^{-1/2}. \quad (5)$$

Eq.(5) can be solved for  $-\dot{x}^i / (g_{lk} \dot{x}^l \dot{x}^k + 2g_{k0} \dot{x}^k + g_{00})^{-1/2}$  and gives

$$\dot{x}^j = -\frac{1}{m}(g_{lk} \dot{x}^l \dot{x}^k + 2g_{k0} \dot{x}^k + g_{00})^{1/2} g^{jk} p_k - g^{jk} g_{k0}. \quad (6)$$

On using Eq.(6), one finds

$$g_{lk}\dot{x}^l\dot{x}^k + 2g_{k0}\dot{x}^k + g_{00} = (g_{00} - g^{il}g_{i0}g_{l0})/(1 - 1/m^2 g^{lk}p_l p_k). \quad (7)$$

By substituting (6) and (7) into (4), one obtains

$$H \simeq \sqrt{p^2 + m^2}(1 + 1/2\gamma_{00}) + 1/2\gamma^{ij}p_i p_j / \sqrt{p^2 + m^2} - p^l \gamma_{l0}. \quad (8)$$

In the presence of electromagnetic fields and in the low velocity limit, the Hamiltonian (8) leads to the Schroedinger equation [23]

$$i\partial\psi(x)/\partial t = [1/2m(p_i - eA_i + m\gamma_{0i})^2 - eA_0 + 1/2m\gamma_{00}]\psi(x). \quad (9)$$

The WFA does not fix the reference frame entirely. The transformations  $x_\mu \rightarrow x_\mu + \xi_\mu$  are still allowed and lead to the "gauge" transformations  $\gamma_{\mu\nu} \rightarrow \gamma_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}$ . In the stationary case the transformations  $\gamma_{00} \rightarrow \gamma_{00}$ ,  $\gamma_{0i} \rightarrow \gamma_{0i} - \xi_{0i}$  leave Eq.(9) invariant. Returning to normal units, the solution of the Schroedinger equation is in this case

$$\psi(x) = \exp\left\{imc/\hbar \int^x \gamma_{0i} dx^i - ie/c\hbar \int^x A_i dx^i\right\} \psi_0(x), \quad (10)$$

where  $\psi_0$  is the solution of the field-free Schroedinger equation. If the electron-lattice interaction is added to Eq.(9), then the resulting equation can be applied to the study of BCS superconductors in weak stationary gravitational fields [24, 25]. This is desirable because BCS superconductors behave in many respects as non viscous fluids. They also exhibit quantization on a macroscopic scale and appear ideally suited to magnify small physical effects.

## 2.2. THE KLEIN-GORDON EQUATION

A well-known form of the the fully covariant Klein-Gordon equation is

$$(g^{\mu\nu}\nabla_\mu\nabla_\nu - m^2)\Phi(x) = 0, \quad (11)$$

where  $\nabla_\mu$  represents covariant differentiation. To first order in the WFA, Eq.(11) becomes

$$[(\eta^{\mu\nu} - \gamma^{\mu\nu})\partial_\mu\partial_\nu - (\gamma^{\alpha\mu} - 1/2\gamma^\sigma_\sigma\eta^{\alpha\mu})_{,\mu}\partial_\alpha]\phi(x) = 0. \quad (12)$$

Eq.(12) has the exact solution [26, 27]

$$\Phi(x) = \exp\{-i\Phi_g\}\phi_0(x) = (1 - i\Phi_g)\phi_0(x), \quad (13)$$

where  $\phi_0(x)$  is the solution of the field-free equation in Minkowski space, and

$$i\Phi_g\phi_0 = \left[\frac{1}{4}\int_P^x dz^\lambda(\gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z))[(x^\alpha - z^\alpha)\partial^\beta - (x^\beta - z^\beta)\partial^\alpha] - \frac{1}{2}\int_P^x dz^\lambda\gamma_{\alpha\lambda}(z)\partial^\alpha\right]\phi_0. \quad (14)$$

Eq.(14) is related to Berry's phase [28]. It is easy to prove by direct substitution that (13) is a solution of (12). In fact

$$\begin{aligned} i\partial_\mu(\Phi_g\phi_0) &= \frac{1}{4}\int_P^x dz^\lambda(\gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z))[\delta_\mu^\lambda\partial^\beta - \delta_\mu^\beta\partial^\alpha]\phi_0(x) + \\ &\quad \frac{1}{4}\int_P^x dz^\lambda(\gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z))[(x^\alpha - z^\alpha)\partial^\beta - \\ &\quad (x^\beta - z^\beta)\partial^\alpha]\partial_\mu\phi_0(x) - \frac{1}{2}\int_P^x dz^\lambda\gamma_{\alpha\lambda}(z)\partial^\alpha\partial_\mu\phi_0(x) - \\ &\quad \frac{1}{2}\gamma_{\alpha\mu}(x)\partial^\alpha\phi_0(x), \end{aligned} \quad (15)$$

from which one gets

$$i\partial_\mu\partial^\mu(\Phi_g\phi_0) = im^2\Phi_g\phi_0 - \gamma_{\mu\alpha}\partial^\mu\partial^\alpha\phi_0 - (\gamma^{\beta\mu} - \frac{1}{2}\gamma^\sigma_\sigma\eta^{\beta\mu})_{,\mu}\partial_\beta\phi_0. \quad (16)$$

The result is proven by substituting (16) into (12). For a closed path in space-time one finds

$$i\Delta\Phi_g\phi_0 = \frac{1}{4}\oint R_{\mu\nu\alpha\beta}L^{\alpha\beta}d\tau^{\mu\nu}\phi_0, \quad (17)$$

where  $L^{\alpha\beta}$  is the angular momentum of the particle of mass  $m$  and  $R_{\mu\nu\alpha\beta}$  is the linearized Riemann tensor. The result found is therefore manifestly gauge invariant.

Unlike the case of the Schroedinger equation discussed above, the gravitational fields considered in this section need not be stationary.

Since Eq.(13) is also a solution of the Landau-Ginzburg equation [26], the present results may be applied to the description of charged and neutral superfluids and Bose-Einstein condensates.

Applications of (13) to the detection of gravitational waves can be found in the literature [26, 29].

### 2.3. MAXWELL EQUATIONS

Consider now Maxwell equations

$$\nabla_\nu\nabla^\nu A_\mu - R_{\mu\sigma}A^\sigma = 0, \quad (18)$$

where the electromagnetic field  $A_\mu$  satisfies the condition  $\nabla_\mu A^\mu = 0$ . If the second term in Eq.(18) is negligible, then Maxwell equations in the WFA are

$$\nabla_\nu \nabla^\nu A_\mu \simeq (\eta^{\sigma\alpha} - \gamma^{\sigma\alpha}) A_{\mu,\alpha\sigma} + R_{\mu\sigma} A^\sigma - (\gamma_{\sigma\mu,\nu} + \gamma_{\sigma\nu,\mu} - \gamma_{\mu\nu,\sigma}) A^{\sigma,\nu} = 0, \quad (19)$$

where use has been made of the Lanczos-DeDonder gauge condition

$$\gamma_{\alpha\nu,\nu} - \frac{1}{2} \gamma_{\sigma,\alpha}^\sigma = 0. \quad (20)$$

Eq.(19) has the solution [27]

$$\begin{aligned} A_\mu(x) = & a_\mu(x) - \frac{1}{4} \int_P^x dz^\lambda (\gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z)) [(x^\alpha - z^\alpha) \partial^\beta a_\mu(x) - \\ & (x^\beta - z^\beta) \partial^\alpha a_\mu(x)] + \frac{1}{2} \int_P^x dz^\lambda \gamma_{\alpha\lambda}(z) \partial^\alpha a_\mu(x) + \\ & \frac{1}{2} \int_P^x dz^\lambda (\gamma_{\beta\mu,\lambda}(z) + \gamma_{\beta\lambda,\mu}(z) - \gamma_{\mu\lambda,\beta}(z)) a^\beta(x), \end{aligned} \quad (21)$$

where  $\partial_\nu \partial^\nu a_\mu = 0$  and  $\partial^\nu a_\nu = 0$ . Eq.(21) can also be written in the form  $A_\mu = \exp(-i\xi) a_\mu$ , where

$$\begin{aligned} \xi = & -\frac{1}{4} \int_P^x dz^\lambda (\gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z)) J^{\alpha\beta} + \\ & \frac{1}{2} \int_P^x dz^\lambda \gamma_{\alpha\lambda}(z) k^\alpha - \frac{1}{2} \int_P^x dz^\lambda \gamma_{\alpha\beta,\lambda}(z) T^{\alpha\beta}, \end{aligned} \quad (22)$$

$J^{\alpha\beta} = L^{\alpha\beta} + S^{\alpha\beta}$  is the total angular momentum,  $(S^{\alpha\beta})^{\mu\nu} = -i(g^{\mu\alpha} g^{\nu\beta} - g^{\mu\beta} g^{\nu\alpha})$  is the spin-1 operator,  $(T^{\alpha\beta})^{\mu\nu} \equiv -i\frac{1}{2}(g^{\mu\alpha} g^{\nu\beta} + g^{\mu\beta} g^{\nu\alpha})$  and  $k^\alpha$  is the momentum of the free photon. All spin effects are therefore contained in the  $S^{\alpha\beta}$  and  $T^{\alpha\beta}$  terms. For a closed path one can again find Eq.(17).

#### 2.4. THE GENERALLY COVARIANT DIRAC EQUATION

Some of the most precise experiments in physics involve spin-1/2 particles. They are very versatile tools that can be used in a variety of experimental situations and energy ranges while still retaining essentially a non-classical behaviour. Within the context of general relativity, De Oliveira and Tiomno [30] and Peres [31] conducted comprehensive studies of the fully covariant Dirac equation. The latter takes the form

$$[i\gamma^\mu(x) D_\mu - m] \Psi(x) = 0, \quad (23)$$

where  $D_\mu = \nabla_\mu + i\Gamma_\mu$ ,  $\{\gamma^\mu(x), \gamma^\nu(x)\} = 2g^{\mu\nu}(x)$ , and the spin connection  $\Gamma^\mu$  is defined by  $D_\mu\gamma_\nu(x) = \nabla_\mu\gamma_\nu(x) + i[\Gamma_\mu(x), \gamma_\nu(x)] = 0$ . By using the definitions  $\Psi(x) = S\tilde{\Psi}(x)$ ,  $S = \exp(-i \int_P^x dz^\lambda \Gamma_\lambda(z))$  and  $\tilde{\gamma}^\mu(x) = S^{-1}\gamma^\mu(x)S$ , in (23) one finds

$$[i\tilde{\gamma}^\mu(x)\nabla_\mu - m]\tilde{\Psi} = 0. \quad (24)$$

By substituting  $\tilde{\Psi} = [-i\tilde{\gamma}^\alpha(x)\nabla_\alpha - m]\psi'$  into (24), one obtains

$$(g^{\mu\nu}\nabla_\mu\nabla_\nu + m^2)\psi' = 0 \quad (25)$$

which, as shown above, has the WFA solution  $\psi' = \exp(-i\Phi_g)\psi_0$ , where  $\psi_0$  is a solution of the Dirac equation in Minkowski space. It is again possible to show that for a closed path the total phase difference suffered by the Dirac wave function is gauge invariant and is given by  $-\frac{1}{4}\oint R_{\mu\nu\alpha\beta}J^{\alpha\beta}d\tau^{\mu\nu}$ , where the total angular momentum is now  $J^{\alpha\beta} = L^{\alpha\beta} + \sigma^{\alpha\beta}$ ,  $\sigma^{\alpha\beta} = -\frac{1}{2}[\gamma^\alpha, \gamma^\beta]$  and  $\gamma^\beta$  represents a usual, constant Dirac matrix [27].

### 3. Applications

Several applications of solutions (10) and (13) to superconductors, gyroscopy and interferometry can now be discussed.

#### 3.1. SUPERCONDUCTORS

By comparing the Schroedinger equation for superconductors in electromagnetic fields with (9) one can immediately draw the following conclusions [24, 25].

i)  $\vec{\nabla}(A_0 - \frac{1}{2}\frac{mc^2}{e}\gamma_{00}) = 0$ . This means that the gravitational field generates an electric field inside the superconductor, contrary to the gravity-free case ( $\gamma_{00} = 0$ ) that yields  $\vec{E} = 0$ . In principle one could therefore detect a gravitational field by means of the electric field it produces inside the superconductor. If the field is Newtonian, then  $\vec{E} = \frac{mg}{e}$  which is the field Schiff and Barnhill [32] predicted gravity would produce inside normal conductors.

ii)  $B_i + \frac{mc^2}{e}\varepsilon_{ijk}\partial^j\gamma^{0i} = 0$  well inside the superconductor where  $B_i$  is known to vanish in the absence of gravitational fields.

iii) The total flux  $\oint(A_i - \frac{mc^2}{e}\gamma_{0i})dx^i = n\frac{hc}{2e}$  is quantized, rather than just the flux of  $B_i$ . This again signifies that  $\gamma_{0i}$  could be measured if the magnetic field it generates were sufficiently large. When the superconductor rotates  $\gamma_{0i} = (\frac{\vec{\omega} \times \vec{r}}{c})_i$ , one finds  $\vec{B} = \frac{2mc}{e}\vec{\omega}$  which is the London moment of rotating superconductors. This result offers tangible evidence that inertia interacts with a quantum system in ways that are compatible with Einstein's views down to lengths of the order of  $10^{-4}cm$ .

These conclusions only apply to *stationary* gravitational fields. Other examples of gravity-induced electric and magnetic fields are discussed in the literature [25].

### 3.2. ROTATION

Consider for simplicity a square interferometer ABCD of side  $l$  in the (xy)-plane, rotating with angular velocity  $\omega$  about the z-axis. The emission and interference of spinless particles of mass  $m$  take place at A and C respectively. Using the metric

$$ds^2 = (1 - \omega^2 \frac{x^2 + y^2}{c^2})(dx^0)^2 + \frac{2\omega}{c}(ydx - xdy)dx^0 - dx^2 - dy^2 - dz^2, \quad (26)$$

and indicating by  $\wp_1$  the path ABC and by  $\wp_2$  the path ADC, the non-vanishing contributions to  $i\Phi_g\phi_0$  given by Eq.(14) are

$$\begin{aligned} \Delta\chi &= -\frac{1}{2} \int_{A,\wp_1}^C dz^\lambda \gamma_{\alpha\lambda}(z) k^\alpha + \frac{1}{2} \int_{A,\wp_2}^C dz^\lambda \gamma_{\alpha\lambda}(z) k^\alpha = \\ &= -\frac{1}{2} \int (dz^0 \gamma_{10} k^1 + dz^0 \gamma_{20} k^2 + dz^1 \gamma_{01} k^0 + dz^2 \gamma_{02} k^0) = \\ &= \frac{\omega}{2c} \left[ - \int_{0,\wp_1}^{\frac{lc}{v}} dz^0 y k^1 + \int_{\frac{lc}{v}}^{\frac{2lc}{v}} dz^0 x k^2 - \int_{0,\wp_2}^{\frac{lc}{v}} dz^0 x k^2 + \right. \\ &\quad \left. \int_{\frac{lc}{v}}^{\frac{2lc}{v}} dz^0 y k^1 \right] - k^0 \left[ \int_{0,\wp_1}^l y dx - \int_{0,\wp_1}^l x dy + \int_{0,\wp_2}^l x dy - \int_{0,\wp_2}^l y dx \right] = \\ &= \frac{\omega l^2}{c} (k \frac{c}{v} + k^0). \end{aligned} \quad (27)$$

For non-relativistic particles  $k^0 \sim \frac{mc}{\hbar}(1 + \frac{v^2}{2c^2})$ ,  $k \sim \frac{mv}{\hbar}$  and the result is  $\Delta\chi \sim \frac{2ml\omega^2}{\hbar}(1 + \frac{v^2}{8c^2})$ . The first term agrees with the results of several relativistic and non-relativistic approximations. In general one obtains from Eq.(14)

$$\Delta\chi = (\frac{2m}{\hbar} + \frac{\hbar k^2}{2mc^2}) \vec{\omega} \cdot \vec{a}, \quad (28)$$

where  $\vec{a}$  represents the area of the interferometer oriented along its normal [26]. It therefore appears that gyroscopy is completely controlled by the quantum phase (28). One also finds that  $\frac{(\Delta\chi)_{ph}}{(\Delta\chi)_{part}} = \frac{\lambda_c}{(\lambda)_{ph}}$ , where  $\lambda_c$  is the Compton wavelength of the particle circulating in the interferometer. This ratio indicates that particle interferometers are more sensitive than photon interferometers for particle masses  $m > \frac{\hbar\nu_{ph}}{c^2}$ .

On applying (22) to photons, one finds that the time integral part of  $\xi$  yields

$$\begin{aligned} -\frac{1}{4} \int_P^x dz^0 (\gamma_{\alpha 0, \beta} - \gamma_{\beta 0, \alpha}) S^{\alpha \beta} - \frac{1}{2} \int_P^x dz^0 \gamma_{\alpha \beta, 0} T^{\alpha \beta} = \\ -\frac{1}{2} \int_P^x dz^0 \gamma_{i 0, j} S^{ij} = \int dt \omega S_z \end{aligned} \quad (29)$$

which represents the spin-rotation coupling, or Mashhoon effect, for photons [13, 14, 27].

### 3.3. GRAVITATIONAL RED-SHIFT

Two light sources of the same frequency are at distances  $r_A$  and  $r_B$  from the origin at the initial time  $x_1^0$ . They are compared at  $r_A$  at the later time  $x_2^0$ . Neglecting spin effects, the phase difference can be simply obtained from (14) using the closed space-time path in the  $(r, x^0)$ -plane with vertices at  $(r_A, x_1^0)$ ,  $(r_B, x_1^0)$ ,  $(r_B, x_2^0)$ ,  $(r_A, x_2^0)$ . The gravitational field is represented by  $\gamma_{00}(r) = 2\varphi(r)$ , where  $\varphi(r)$  is the Newtonian potential. One finds

$$\begin{aligned} \Delta\chi &= \frac{1}{2} \int_{x_1^0}^{x_2^0} dz^0 [\gamma_{\alpha 0, \beta}(r_B) - \gamma_{\beta 0, \alpha}(r_B)] (x^\alpha - z^\alpha) k^\beta + \\ &\frac{1}{2} \int_{x_2^0}^{x_1^0} dz^0 \gamma_{\alpha 0, \beta}(r_A) (x^\alpha - z^\alpha) k^\beta - \frac{1}{2} \int_{x_1^0}^{x_2^0} dz^0 \gamma_{\alpha 0}(r_B) k^\alpha - \\ &\frac{1}{2} \int_{x_2^0}^{x_1^0} dz^0 \gamma_{\alpha 0}(r_A) k^\alpha = \\ &-\frac{k^0}{2} (x_2^0 - x_1^0) [\gamma_{00}(r_B) - \gamma_{00}(r_A)] - \\ &\frac{k^0}{4} (x_2^0 - x_1^0)^2 [\gamma_{00,1}(r_B) + \gamma_{00,1}(r_A)]. \end{aligned} \quad (30)$$

The first term gives the usual red-shift formula  $(\frac{\Delta\nu}{\nu})_1 = -\frac{1}{c^2} [\varphi(r_B) - \varphi(r_A)]$ . The second term yields the additional correction  $(\frac{\Delta\nu}{\nu})_2 = -\frac{x_2^0 - x_1^0}{4} [\gamma_{00,1}(r_B) + \gamma_{00,1}(r_A)]$ . In an experiment of the type carried out by Pound and Rebka the ratio of the two terms is  $(\frac{\Delta\nu}{\nu})_2 / (\frac{\Delta\nu}{\nu})_1 \simeq \frac{2l}{R_\oplus}$ , where  $l = r_B - r_A$ . The second term  $(\frac{\Delta\nu}{\nu})_2$  should therefore be measurable for sufficiently high values of  $l$ .

### 3.4. SCHWARZSCHILD METRIC

If Earth is assumed perfectly spherical and homogeneous and rotation is neglected, then its gravitational field can be described by the Schwarzschild



metric[33]

$$ds^2 = \left(1 - \frac{2M_\oplus}{r}\right)(dx^0)^2 - \left(1 - \frac{2M_\oplus}{r}\right)^{-1}dr^2 - r^2d\theta^2 - r^2\sin\theta^2d\varphi^2. \quad (31)$$

Assuming, for simplicity that a square interferometer of side  $l$  is placed in a vertical plane at one of the poles and that particle emission and interference occur at opposite corners, one finds

$$\Delta\chi = \frac{GM_\oplus l^2 m}{R_\oplus^2 \hbar v} \left( 1 + \frac{3v^2}{2c^2} - \frac{3l}{2R_\oplus} - \frac{3v^2 l}{4c^2 R_\oplus} \right). \quad (32)$$

The first term in Eq.(32) is the term observed with a neutron interferometer in the well known COW experiment [2]. If  $v \sim 10^{-5}c$ , then the De Broglie's wavelength for neutrons is  $\sim 10^{-8}cm$ . General relativity appears therefore to be valid down to lengths of this order of magnitude. The ratio of the terms on the r.h.s. of (32) is  $1 : 10^{-10} : 10^{-7} : 10^{-17}$ . The last term is extremely small and may be neglected. Of the remaining terms, the second represents a special relativistic correction, which is smaller than the general relativity effect represented by the third term. The ratio of the latter to the magnitude of the Sagnac effect also is  $\sim 10^{-7}$  and appears difficult to observe at present.

### 3.5. LENSE-THIRING FIELD OF EARTH

The non-vanishing components of  $\gamma_{\mu\nu}$  are in this instance [34]

$$\begin{aligned} \gamma_{00} &= \gamma_{11} = \gamma_{22} = \gamma_{33} = \frac{2GM_\oplus}{c^2 r} \\ \gamma_{01} &= \frac{4GM_\oplus \omega a^2 (y + y')}{5c^3 r^3}, \gamma_{02} = \frac{4GM_\oplus \omega a^2 (x + x')}{5c^3 r^3}, \end{aligned} \quad (33)$$

where  $r^2 = (x + x')^2 + (y + y')^2 + (z + z')^2$ , Earth is again assumed spherical and homogeneous,  $\omega$  its angular velocity about the  $z'$ -axis, and  $(x', y', z')$  are the coordinates at the point  $A$  at which the interferometer beam is split in the coordinate system with origin at the centre of the Earth. Interference occurs at the opposite vertex  $C$ . The frame  $z^\mu$  has origin at  $A$ , is at rest relative to  $z'^\mu$  and the plane of the interferometer is chosen for simplicity to coincide with the  $(x, y)$ -plane and parallel to the  $(x', y')$ -plane. The time at which the particle beam is split at  $A$  is  $z^0 = z'^0 = 0$  [35]. If, in particular,  $A$  coincides with a pole, then  $\Delta\chi = \frac{2G}{c^2 R_\oplus^3} J_\oplus \frac{ml^2}{\hbar}$ , where  $J_\oplus = \frac{2M_\oplus R_\oplus^2 \omega}{5}$  is the angular momentum of Earth. Taking into account that the precession frequency of a gyroscope in orbit is  $\Omega = \frac{GJ_\oplus}{2c^2 R_\oplus^3}$ , one can also write  $\Delta\chi = \Omega\Pi$ ,

where  $\Pi = \frac{4ml^2}{\hbar}$  replaces the period of a satellite in the classical calculation. Its value,  $\Pi \sim 1.4 \times 10^8 s$  for neutron interferometers with  $l \sim 10^2 cm$ , is rather high and yields  $\Delta\chi \sim 10^{-7} rad$ . This suggests that the development and use of heavy particle interferometers would be particularly advantageous in attempts to measure the Lense-Thirring effect.

#### 4. Helicity precession of fermions

Consider the line element  $ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$  and the set of tangent vectors  $\vec{e}_\mu = \partial_\mu \vec{P}$  that forms the coordinate basis that spans the manifold  $g_{\mu\nu}(x) = \partial_\mu \vec{P} \cdot \partial_\nu \vec{P} \equiv \vec{e}_\mu \cdot \vec{e}_\nu$ . The principle of equivalence ensures the existence of an orthonormal tetrad frame  $\vec{e}_{\hat{\mu}} = \partial_{\hat{\mu}} \vec{P}$  such that for a local tangent space defined at any given point of space-time  $\eta_{\hat{\mu}\hat{\nu}} = \vec{e}_{\hat{\mu}} \cdot \vec{e}_{\hat{\nu}}$ . The principle underlying the tetrad formalism therefore requires that for a sufficiently small region of space-time  $\vec{e}_\mu$  be mapped onto  $\vec{e}_{\hat{\mu}}$  using a set of projection functions  $e_\mu^\nu$  and their inverses  $e_{\hat{\mu}}^\nu$  such that

$$\begin{aligned} \vec{e}_{\hat{\mu}} &= e_{\hat{\mu}}^\nu \vec{e}_\nu, \vec{e}_\mu = e_\mu^{\hat{\nu}} \eta_{\hat{\mu}\hat{\nu}} = e_{\hat{\mu}}^\mu e_\nu^\mu g_{\mu\nu}(x), \\ e_\mu^\nu e_{\hat{\mu}}^{\hat{\alpha}} &= \delta_{\hat{\mu}}^{\hat{\alpha}}, e_\mu^{\hat{\alpha}} e_{\hat{\nu}}^\alpha = \delta_\mu^\alpha. \end{aligned} \quad (34)$$

When  $\vec{e}_\mu$  refers to an observer with acceleration  $\vec{a}$  rotating with angular velocity  $\vec{\omega}$ , one finds [9]

$$ds^2 = [(1 + \vec{a} \cdot \vec{x})^2 + (\vec{\omega} \cdot \vec{x})^2 - \omega^2 x^2] dx_0^2 - 2dx_0 d\vec{x} \cdot (\vec{\omega} \times \vec{x}) - d\vec{x} \cdot d\vec{x}, \quad (35)$$

while

$$\begin{aligned} \vec{e}_{\hat{0}} &= (1 + \vec{a} \cdot \vec{x})^{-1} [\vec{e}_0 - (\vec{\omega} \times \vec{x})^k \vec{e}_k], \vec{e}_{\hat{i}} = \vec{e}_i, \\ e_{\hat{0}}^0 &= (1 + \vec{a} \cdot \vec{x})^{-1}, e_{\hat{0}}^k = -(1 + \vec{a} \cdot \vec{x})^{-1} (\vec{\omega} \times \vec{x})^k, \\ e_{\hat{0}}^{\hat{0}} &= 1 + \vec{a} \cdot \vec{x}, e_{\hat{0}}^{\hat{k}} = (\vec{\omega} \times \vec{x})^k, e_{\hat{i}}^{\hat{0}} = 0, e_{\hat{i}}^{\hat{k}} = \delta_i^k. \end{aligned} \quad (36)$$

Also, from  $D_\mu \gamma_\nu(x) = 0$  and  $\gamma_\mu(x) = e_{\hat{\mu}}^\mu \gamma_{\hat{\mu}}$ , where  $\gamma_{\hat{\mu}}$  represents the usual Dirac matrices, one finds  $\Gamma_\mu(x) = \frac{1}{4} \sigma^{\alpha\beta} \Gamma_{\alpha\hat{\beta}\mu} e_{\hat{\mu}}^\mu$ . The Ricci coefficients are  $\Gamma_{\nu\hat{\alpha}\hat{\beta}} = \frac{1}{2} (C_{\nu\hat{\alpha}\hat{\beta}} + C_{\alpha\hat{\beta}\nu} - C_{\beta\hat{\nu}\alpha})$  and  $C_{\nu\hat{\alpha}\hat{\beta}} = \eta_{\mu\nu} e_{\hat{\alpha}}^\alpha e_{\hat{\beta}}^\beta (\partial_\alpha e_{\hat{\beta}}^\mu - \partial_{\hat{\beta}} e_\alpha^\mu)$ . It also follows that  $\Gamma_0 = -\frac{1}{2} a_i \sigma^{0i} - \frac{1}{2} \vec{\omega} \cdot \vec{\sigma}$ ,  $\Gamma_i = 0$ , with  $\sigma^{0i} = \frac{1}{2} [\gamma^0, \gamma^i]$ . The Hamiltonian is obtained by isolating the time derivative in the Dirac equation. The result is

$$\begin{aligned} H &= \vec{\alpha} \cdot \vec{p} + m\beta + V(x) \\ V(x) &= \frac{1}{2} [(\vec{a} \cdot \vec{x})(\vec{p} \cdot \vec{\alpha}) + (\vec{p} \cdot \vec{\alpha})(\vec{a} \cdot \vec{x})] + m\beta(\vec{a} \cdot \vec{x}) - \vec{\omega} \cdot (\vec{L} + \frac{\vec{\sigma}}{2}), \end{aligned} \quad (37)$$

where  $\vec{L}$  is the orbital angular momentum and  $\vec{\sigma}$  are the usual Pauli matrices. The first three terms in  $V(x)$  represent relativistic energy-momentum effects. The term  $-\vec{\omega} \cdot \vec{L}$  is a Sagnac-type effect. The last term,  $-\frac{1}{2}\vec{\omega} \cdot \vec{\sigma}$ , is the spin-rotation coupling, or Mashhoon effect. The non-relativistic effects can be obtained by applying three successive Foldy-Wouthysen transformations to  $H$ . One obtains to lowest order

$$H = m\beta + \beta \frac{p^2}{2m} + \beta m(\vec{a} \cdot \vec{x}) + \frac{\beta}{2m} \vec{p}(\vec{a} \cdot \vec{x}) \cdot \vec{p} - \vec{\omega} \cdot (\vec{L} + \frac{\vec{\sigma}}{2}) + \frac{1}{4m} \vec{\sigma} \cdot (\vec{a} \times \vec{p}). \quad (38)$$

The third term in Eq.(38) is the energy-momentum effect observed by Bonse and Wroblewski [4]. The term  $-\vec{\omega} \cdot \vec{L}$  was predicted by Page [36] and observed by Werner and collaborators [3]. The term  $-\vec{\omega} \cdot \frac{\vec{\sigma}}{2}$  was found by Mashhoon. Hehl and Ni [9] re-derived all terms and also predicted the existence of the fourth term (a kinetic energy effect) and of the last term (spin-orbit coupling). Equations (37) and (38) can also be obtained by isolating the quantum phase in the wave function. The spin-rotation coupling term deserves a few comments. As discussed by Mashhoon, the effect violates the hypothesis of locality, namely that an accelerated observer is locally equivalent to an instantaneously comoving observer. This hypothesis is valid for classical point-like particles and optical rays and is widely used in relativity. The effect also violates the equivalence principle because it does not couple universally to matter [37]. No direct experimental verification of the Mashhoon effect has so far been reported, though the data given in [19] can be re-interpreted as due to the coupling of Earth's rotation to the nuclear spins of mercury. The effect is also consistent with a small depolarization of electrons in storage rings [38]. It is shown below that it plays an essential role in measurements of the anomalous magnetic moment, or  $g - 2$  factor, of the muon.

#### 4.1. SPIN-ROTATION COUPLING IN MUON $G-2$ EXPERIMENTS

Precise measurements of the  $g - 2$  factor involve muons in a storage ring consisting of a vacuum tube, a few meters in diameter, in a uniform, vertical magnetic field  $\vec{B}$ . Muons on equilibrium orbits within a small fraction of the maximum momentum are almost completely polarized with spin vectors pointing in the direction of motion. As the muons decay, the highest energy electrons with spin almost parallel to the momentum, are projected forward in the muon rest frame and are detected around the ring. Their angular distribution does therefore reflect the precession of the muon spin along the cyclotron orbits [39, 40]. Let us start from the covariant Dirac equation (23). It is convenient to use the chiral representation for the usual Dirac

matrices

$$\begin{aligned}\gamma^0 &= \beta = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \alpha^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}, \\ \sigma^{0i} &= i \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}, \sigma^{ij} = \epsilon_{ij}^k \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}, \gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.\end{aligned}\quad (39)$$

One must now add to the the Hamiltonian the effect of a magnetic field  $\vec{B}$  on the total (magnetic plus anomalous) magnetic moment of the particle. Assuming for simplicity that all quantities in  $H$  are time-independent and referring them to a left-handed tern of axes comoving with the particle in the  $x_3$ -direction and rotating in the  $x_2$ -direction, one finds

$$\begin{aligned}H &= \alpha^3 p_3 + m\beta + \frac{1}{2}[-a_1 R(\alpha^3 p_3) - (\alpha^3 p_3)a_1 R] + \beta m a_1 R - \vec{\omega} \cdot \vec{L} - \\ &\quad \frac{1}{2}\omega_2 \sigma^2 + \mu B \sigma^2 \equiv H_0 + H',\end{aligned}\quad (40)$$

where  $B_2 = -B$ ,  $\mu = (1 + \frac{g-2}{2})\mu_0$ ,  $\mu_0 = \frac{e\hbar}{2mc}$  is the Bohr magneton,  $H' = -\frac{1}{2}\omega_2 \sigma^2 + \mu\beta \sigma^2$  and  $R$  is the radius of the muon's orbit. Electric fields used to stabilize the orbits and stray radial electric fields can also affect the muon spin. Their effects can be cancelled by choosing an appropriate muon momentum and will be neglected in what follows. Before decay the muon states can be represented as

$$|\psi(t)\rangle = a(t)|\psi_+\rangle + b(t)|\psi_-\rangle, \quad (41)$$

where  $|\psi_+\rangle$  and  $|\psi_-\rangle$  are the right and left helicity states of  $H_0$ . Substituting (41) into the Schroedinger equation  $i\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle$ , one obtains

$$\begin{aligned}i\frac{\partial}{\partial t} \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} \langle \psi_+ | (H_0 + H') | \psi_+ \rangle & \langle \psi_+ | H' | \psi_- \rangle \\ \langle \psi_- | H' | \psi_+ \rangle & \langle \psi_- | (H_0 + H') | \psi_- \rangle \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\ &= \begin{pmatrix} E - i\frac{\Gamma}{2} & i(\frac{\omega_2}{2} - \mu B) \\ -i(\frac{\omega_2}{2} - \mu B) & E - i\frac{\Gamma}{2} \end{pmatrix} \equiv M \begin{pmatrix} a \\ b \end{pmatrix},\end{aligned}\quad (42)$$

where  $\Gamma$  represents the width of the muon. Notice that the spin-rotation coupling is off diagonal in (42). This is a clear indication that the Mashhoon effect violates the equivalence principle [37]. The matrix  $M$  can be diagonalized. Its eigenvalues are  $h_1 = E - i\frac{\Gamma}{2} + (\frac{\omega_2}{2} - \mu B)$ ,  $h_2 = E - i\frac{\Gamma}{2} - (\frac{\omega_2}{2} - \mu B)$ , with the corresponding eigenvectors

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}[i|\psi_+\rangle + |\psi_-\rangle]; |\psi_2\rangle = \frac{1}{\sqrt{2}}[-i|\psi_+\rangle + |\psi_-\rangle]. \quad (43)$$

The solution of Eq.(42) is therefore

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}(e^{-ih_1t}|\psi_1\rangle + e^{-ih_2t}|\psi_2\rangle) = \frac{1}{2}[(ie^{-ih_1t} - ie^{-ih_2t})|\psi_+\rangle + (e^{-ih_1t} + e^{-ih_2t})|\psi_-\rangle], \quad (44)$$

where  $|\psi(0)\rangle = |\psi_-\rangle$ . The spin-flip probability is

$$P_{\psi_-\rightarrow\psi_+} = |\langle\psi_+|\psi\rangle|^2 = \frac{e^{-\Gamma t}}{2}[1 - \cos(2\mu B - \omega_2)t], \quad (45)$$

where the  $\Gamma$ -term accounts for the observed exponential decrease in electron counts due to the loss of muons by radioactive decay [40]. The spin-rotation contribution to  $P_{\psi_-\rightarrow\psi_+}$  is represented by  $\omega_2$  which is the cyclotron angular velocity  $\frac{eB}{m}$ . The spin-flip angular frequency is then

$$\Omega = 2\mu B - \omega_2 = (1 + \frac{g-2}{2})\frac{eB}{m} - \frac{eB}{m} = \frac{g-2}{2}\frac{eB}{m}, \quad (46)$$

which is precisely the observed modulation frequency of the electron counts [41](see also Fig. 19 of Ref.[40]). This result is independent of the value of the anomalous magnetic moment of the particle. It is therefore the Mashhoon effect that gives prominence to the  $g-2$  term in  $\Omega$  by exactly cancelling, in  $2\mu B$ , the much larger contribution  $\mu_0$  that comes from fermions with no anomalous magnetic moment [42].

It is perhaps odd that spin-rotation coupling as such has almost gone unnoticed for such a long time. It is however significant that its effect is observed in an experiment that has already provided crucial tests of quantum electrodynamics and a test of Einstein's time-dilation formula to better than a 0.1 percent accuracy. Recent versions of the experiment [43, 44, 45] have improved the accuracy of the measurements from 270 ppm to 1.3 ppm. This bodes well for the detection of effects involving spin, inertia and electromagnetic fields, or inertial fields to higher order.

## 5. Neutrino Oscillations

Neutrino beams produced in weak interactions may be considered as a superposition of different mass eigenstates. As a beam propagates, different components of the beam evolve differently so that the probability of finding different eigenstates in the beam varies with time. The consequences of this can be explored in a number of cases.

### 5.1. NEUTRINO HELICITY OSCILLATIONS

Let us consider a beam of high energy neutrinos. If the neutrino source rotates, the effective Hamiltonian for the mass eigenstates can be written

as  $H_e = (p^2 + m^2)^{\frac{1}{2}} + \Gamma_0 \approx p + \frac{m^2}{2E} - \frac{1}{2}\vec{\omega} \cdot \vec{\sigma}$ . For simplicity, consider a one-generation of neutrino that can now be written as a superposition of  $\nu_L$  and  $\nu_R$  in the form

$$|\nu(t)\rangle = a_L(t)|\nu_L\rangle + b_R(t)|\nu_R\rangle. \quad (47)$$

It is well known that the standard model contemplates only the existence of  $\nu_L$ , while  $\nu_R$  is considered sterile and therefore unobservable. Strictly speaking one should consider the helicity states  $\nu_{\pm}$  (that are mass eigenstates) in (47), however at high energies  $\nu_L \simeq \nu_-$ ,  $\nu_R \simeq \nu_+$ . Assuming that  $m_1 \neq m_2$ , taking  $p_1 \sim p_2$  along the  $x_3$ -axis and substituting (47) into the Schroedinger equation that corresponds to  $H_e$ , one obtains

$$i\frac{\partial}{\partial t} \begin{pmatrix} a_L \\ b_R \end{pmatrix} = \begin{pmatrix} p + \frac{m_1^2}{2E} & -\frac{\omega_1}{2} - i\frac{\omega_2}{2} \\ -\frac{\omega_1}{2} + i\frac{\omega_2}{2} & p + \frac{m_2^2}{2E} \end{pmatrix} \begin{pmatrix} a_L \\ b_R \end{pmatrix} \equiv M_{12} \begin{pmatrix} a_L \\ b_R \end{pmatrix}. \quad (48)$$

The eigenvalues of  $M_{12}$  are

$$k_{\mp} = p + \frac{m_1^2 + m_2^2}{4E} \mp [(\frac{\Delta m^2}{2E})^2 + \omega_{\perp}^2]^{\frac{1}{2}}, \quad (49)$$

where  $\Delta m^2 \equiv m_1^2 - m_2^2$ , and  $\omega_{\perp}^2 \equiv \omega_1^2 + \omega_2^2$ . The eigenvectors are

$$|\nu_1\rangle = b_1[\eta_1|\nu_L\rangle + |\nu_R\rangle], |\nu_2\rangle = b_2[\eta_2|\nu_L\rangle + |\nu_R\rangle], \quad (50)$$

where

$$\eta_1 = \frac{\omega_1 + i\omega_2}{\Omega + \frac{\Delta m^2}{2E}}, \eta_2 = \frac{\omega_1 + i\omega_2}{-\Omega + \frac{\Delta m^2}{2E}}, \quad (51)$$

$$|b_1|^2 = \frac{1}{1 + |\eta_1|^2}, |b_2|^2 = \frac{1}{1 + |\eta_2|^2}, \quad (52)$$

and

$$\Omega = \left[ \left( \frac{\Delta m^2}{2E} \right)^2 + \omega_{\perp}^2 \right]^{\frac{1}{2}}. \quad (53)$$

One therefore finds

$$|\nu(t)\rangle = \frac{b_1}{\eta_1 - \eta_2} \exp \left[ -i \left( p + \frac{m_1^2 + m_2^2}{4E} \right) t \right] \times \left[ \left( e^{i\frac{\Omega}{2}t} \eta_1 - e^{-i\frac{\Omega}{2}t} \eta_2 \right) \nu_L + 2i \sin\left(\frac{\Omega t}{2}\right) \nu_R \right], \quad (54)$$

where the initial condition is  $\nu(0) = \nu_L$ . One obtains the transition probability

$$P_{\nu_L \rightarrow \nu_R} = |\langle \nu_R | \nu(t) \rangle|^2 = \frac{\omega_{\perp}^2}{2\Omega^2} [1 - \cos(\Omega t)]. \quad (55)$$

If the neutrinos have mass, then the magnitude of the transition probability becomes appreciable if

$$\omega_{\perp} \geq \frac{\Delta m^2}{2E}. \quad (56)$$

Unlike the flavour oscillations generated by the MSW-mechanism [46, 47] that require  $\Delta m^2 \neq 0$ , helicity oscillations can occur also when  $m_1 = m_2$  and  $m_1 = m_2 = 0$ . They are interesting because  $\nu_R$ 's, if they exist, do not interact with matter and would therefore provide an energy dissipation mechanism with possible astrophysical implications. The conversion rate that  $\nu_L \rightarrow \nu_R$  is not large for galaxies and white dwarfs. Assume in fact that  $\omega_{\perp} \gg \Delta m^2/2E$  and that the beam of neutrinos consists of  $N_L(0)$  particles at  $z = 0$ . One immediately obtains from (55) that the relative numbers of particles at  $z = 0$  are [48]

$$N_L(z) = N_L(0) \cos^2 \left( \frac{\omega_{\perp} z}{2c} \right), N_R(z) = N_L(0) \sin^2 \left( \frac{\omega_{\perp} z}{2c} \right). \quad (57)$$

One then obtains from (57)  $N_R \sim 10^{-6} N_L(0)$  for galaxies of typical size  $L$  such that  $\omega_{\perp} L \sim 200 km/s$ . Similarly, for white dwarfs  $\omega_{\perp} \sim 1.0 s^{-1}$  and one finds  $N_R \sim 10^{-4} N_L(0)$ . In the case of the Sun  $\omega_{\perp} \sim 7.3 \times 10^{-5} - 2.4 \times 10^{-6} s^{-1}$  and the conversion rate peaks at distances  $L \sim 10^{15} - 4 \times 10^{16} cm$ , well in excess of the average Sun-Earth distance. Helicity oscillations could not therefore explain the solar neutrino puzzle without additional assumptions about the Sun's structure [49]. For neutron stars, however, the dynamics of the star could be affected by this cooling mechanism. In fact neutrinos diffuse out of a canonical neutron star in a time 1 to 10s, during which they travel a distance  $3 \times 10^9 cm$  between collisions. At distances  $L \sim 5 \times 10^6 cm$  (the star's radius) the conversion rate is  $N_R \sim 0.5 N_0$ . Even higher cooling rates may occur at higher rotational speeds and prevent the formation of a pulsar. These results do not require the existence of a magnetic moment for the neutrino (which would also require some mass). Its effect could be taken into account by adding the term  $\vec{\mu} \cdot \vec{B}$  to  $H_e$ . In all instances considered, however, magnetic spin-flip rates of magnitude comparable to those discussed would require neutrino magnetic moments vastly in excess of the value  $10^{-19} \mu_0 (\frac{m_{\nu}}{1 eV})$  predicted by the standard  $SU(2) \times U(1)$  electroweak theory [50].

## 5.2. HELICITY OSCILLATIONS IN A MEDIUM

The behaviour of neutrinos in a medium is modified by a potential  $V$ . When this is taken into account, the effective Hamiltonian becomes (after subtracting from the diagonal terms a common factor which contributes

only to the overall phase)

$$H = p + \frac{m^2}{2E} - V - \frac{1}{2}\vec{\omega} \cdot \vec{\sigma}. \quad (58)$$

For simplicity, consider again a one-generation of neutrino and assume that  $V$  is constant. Applying the diagonalization procedure of the previous section to the new Hamiltonian, leads to the transition probability

$$P_{\nu_L \rightarrow \nu_R} = \frac{\omega_\perp^2}{2\Omega'^2} [1 - \cos(\Omega't)], \quad (59)$$

where  $\Omega' = [(V + \frac{\Delta m^2}{2E})^2 + \omega_\perp^2]^{\frac{1}{2}}$ . One finds from (59) that spin-flip transitions are strongly suppressed when  $V + \frac{\Delta m^2}{2E} > \omega_\perp$  and only the  $\nu_L$  component is present in the beam. If  $\omega_\perp > V + \frac{\Delta m^2}{2E}$ , then the  $\nu_L$  flux has effective modulation. Resonance occurs at  $V = \frac{-\Delta m^2}{2E}$ . Consider now the rotating core of a supernova. In this case  $V$  can be relatively large, of the order of several electron volts and corresponds to the interaction of neutrinos with the particles of the medium. For right-handed neutrinos  $V$  vanishes. Assuming that the star does not radiate more energy as  $\nu_R$ 's than as  $\nu_L$ 's, one finds  $L_{\nu_L} \sim L_{\nu_R} \sim 5 \times 10^{53} \text{ erg/s}$ . As the star collapses, spin-rotation coupling acts on both  $\nu_L$  and  $\nu_R$ . The  $\nu_L$ 's become trapped and leak toward the exterior ( $l \sim 1.5 \times 10^7 \text{ cm}$ ), while their interaction with matter is  $V \sim 14(\rho/\rho_c) \text{ eV}$  and increases therefore with the medium's density, which at the core is  $\rho_c \sim 4 \times 10^{14} \text{ g/cm}^3$ . The  $\nu_R$ 's escape. As  $\rho$  increases, the transition  $\nu_L \rightarrow \nu_R$  is inhibited (off resonance). One also finds  $\frac{\Delta m^2}{2E} < \omega_\perp$  when  $\Delta m^2 < 10^{-5} \text{ eV}^2$ ,  $E \sim 10 \text{ MeV}$ ,  $\omega_\perp \sim 6 \times 10^3 \text{ s}^{-1}$ . It then follows from (59) that

$$L_{\nu_L} \sim L_{\nu_R} \sin^2\left(\frac{\omega_\perp l}{2c}\right), \quad (60)$$

where  $\frac{\omega_\perp l}{2c} \sim \frac{\pi}{2}$ . In the time  $\frac{l}{c} \sim 5 \times 10^{-4} \text{ s}$ , the energy associated with the  $\nu_R \rightarrow \nu_L$  conversion is  $\sim 2.5 \times 10^{50} \text{ erg}$  which is just the missing energy required to blow up the mantle of the collapsing star [48].

### 5.3. NEUTRINO FLAVOUR OSCILLATIONS

Consider a beam of neutrinos of fixed energy  $E$  emitted at point  $(r_A, t_A)$  of the  $(r, t)$ -plane. Assume also that the particles are in a weak flavour eigenstate that is a linear superposition of mass eigenstates  $m_1$  and  $m_2$ , with  $m_1 \neq m_2$ . It is argued in the literature [51, 52] that if interference is observed at the same space-time point  $(r_B, t_B)$ , then the lighter component must have left the source at a later time  $\Delta t = \frac{r_B - r_A}{v_1} - \frac{r_B - r_A}{v_2}$ , where  $v_1$  and  $v_2$  are the velocities of the eigenstates of masses  $m_1$  and  $m_2$  respectively.



Because of the difference in travel time  $\Delta t$ , gravity induced neutrino flavour oscillations will ensue even though gravity couples universally to matter. Ignoring spin contributions, the phase difference of the two mass eigenstates can be calculated from (14) in a completely gauge invariant way. Assume the neutrinos propagate in a gravitational field described by the Schwarzschild metric. When the closed space-time path in (14) is extended to the triangle  $(r_A, t_A), (r_B, t_B), (r_A, t_A + \Delta t)$ , one obtains

$$\begin{aligned} (i\Delta\Phi_g\phi_0)_{m_1} - (i\Delta\Phi_g\phi_0)_{m_2} = \\ \frac{r_g E}{2} \left[ -\frac{v_1 \Delta t}{2} - \frac{1}{v_1} \ln \left( \frac{r_B}{r_A + v_1 \Delta t} \right) + \left( -v_1 + \frac{1}{v_2} + v_2 \right) \ln \frac{r_B}{r_A} \right] \simeq \\ \frac{r_g E}{2} \left( \frac{1}{v_2} - \frac{1}{v_1} + v_2 - v_1 \right) \ln \frac{r_B}{r_A}, \end{aligned} \quad (61)$$

where the approximation  $v_1 \Delta t \ll r_A$  has been used in deriving the last result. On using the equation  $1/v = E/p$  and the approximations  $v \sim 1 - \frac{m^2}{2E^2} - \frac{m^4}{8E^4}$ ,  $1/v \sim 1 + \frac{m^2}{2E^2} + \frac{3m^4}{8E^4}$ , one arrives at the final result

$$\begin{aligned} (i\Delta\Phi_g\phi_0)_{m_1} - (i\Delta\Phi_g\phi_0)_{m_2} \simeq \\ \frac{MGc^5}{4\hbar E^3} (m_2^4 - m_1^4) \ln \frac{r_B}{r_A} = \\ 1.37 \times 10^{-19} \left( \frac{M}{M_\odot} \right) \left( \frac{\Delta m^4}{eV^4} \right) \left( \frac{MeV}{E} \right)^3 \ln \frac{r_B}{r_A}. \end{aligned} \quad (62)$$

The effect therefore exists, but is extremely small in typical astrophysical applications.

Torsion-induced neutrino oscillations have been considered by de Sabata and collaborators [53, 54].-

#### 5.4. THE EQUIVALENCE PRINCIPLE AND NEUTRINO OSCILLATIONS

Gravitational fields can not generate neutrino oscillations if gravity couples universally to matter. As first pointed out by Gasperini [55], violations of the equivalence principle could in principle affect the behaviour of neutrinos and be tested in experiments on neutrino oscillations [56]. Consider a spinless particle in a Newtonian gravitational field. Its Hamiltonian in the WFA is

$$H = (1 - \gamma_{00})^{\frac{1}{2}} (p^2 - \gamma^{ij} p_i p_j + m^2)^{\frac{1}{2}} - p_i \gamma^{i0}, \quad (63)$$

which, in the simple case of a Newtonian potential, becomes

$$H \sim p + \frac{m^2}{2p} - \frac{1}{2} p \gamma_{00}, \quad (64)$$

where  $\gamma_{00} = 2\alpha\varphi(r)$  and  $\alpha = 1$  if the principle of equivalence is not violated. Deviations from the equivalence principle are parameterized by assuming that  $\alpha \neq 1$  and takes different values for different neutrino mass eigenstates. Assume the mass eigenstates  $\nu_1, \nu_2$  are related to the weak eigenstates by the transformation  $\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = U \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}$ , where the unitary matrix  $U = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$  represents mixing in the two-generation case. The weak eigenstates then evolve according to the equation

$$i \frac{\partial}{\partial t} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = U^\dagger \left( p + \frac{m^2}{2E} - \alpha E \right) U \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}, \quad (65)$$

where  $m^2 = \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix}$ , and  $\alpha = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$ . One therefore finds

$$i \frac{\partial}{\partial t} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = M_W \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}, \quad (66)$$

$$M_W = \begin{pmatrix} E + \left( \frac{m_1^2}{2E} - E\varphi\alpha_1 \right) \cos^2 \theta + & \left( \frac{\Delta m^2}{4E} - E\varphi \frac{\Delta\alpha}{2} \right) \sin(2\theta) \\ \left( \frac{m_2^2}{2E} - E\varphi\alpha_2 \right) \sin^2 \theta & E + \left( \frac{m_1^2}{2E} - E\varphi\alpha_1 \right) \sin^2 \theta + \\ \left( \frac{\Delta m^2}{4E} - E\varphi \frac{\Delta\alpha}{2} \right) \sin(2\theta) & \left( \frac{m_2^2}{2E} - E\varphi\alpha_2 \right) \cos^2 \theta \end{pmatrix}. \quad (67)$$

Since the overall phase is unobservable, subtracting the constant  $E + (m_1^2 - E\varphi\alpha_1) \sin^2 \theta + (m_2^2 - E\varphi\alpha_2) \cos^2 \theta$  from the diagonal terms of  $M_W$  does not affect oscillations in which only the relative phases of the mass eigenstates are involved. This leads to the equation

$$i \frac{\partial}{\partial t} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \frac{1}{2} \left( \frac{\Delta m^2}{2E} - E\varphi\Delta\alpha \right) \begin{pmatrix} -2\cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & 0 \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}. \quad (68)$$

The solutions of (68) are

$$\begin{aligned} \nu_e(t) &= C_1 e^{-i\omega t} + C_2 e^{i\omega t} \\ \nu_\mu(t) &= D_1 e^{-i\omega t} + D_2 e^{i\omega t}, \end{aligned} \quad (69)$$

with the condition  $|\nu_e(t)|^2 + |\nu_\mu(t)|^2 = 1$ . One finds  $\omega = \frac{\Delta m^2}{4E} - E\varphi \frac{\Delta\alpha}{2}$ . The initial condition  $\nu_e(0) = 1$  is also used to determine the constants in (69).

One finds  $C_1 = \sin^2 \theta$ ,  $C_2 = \cos^2 \theta$ ,  $D_1 = \sin \theta \cos \theta$ ,  $D_2 = -\sin \theta \cos \theta$ . The transition probability therefore is

$$P_{\nu_e \rightarrow \nu_\mu} = \sin^2(2\theta) \sin^2(\omega t). \quad (70)$$

In the absence of gravity,  $\alpha_1 = \alpha_2 = 0$ , flavour oscillations in vacuum occur according to the MSW mechanism and are driven by  $\Delta m^2$ . The MSW oscillations take place also when  $\alpha_1 = \alpha_2 \neq 0$ . On the other hand, when gravity is present and  $\alpha_1 \neq \alpha_2 \neq 0$ , flavour oscillations occur not only if  $\Delta m^2 \neq 0$ , but also when  $\Delta m^2 = 0$ , with either  $m_1 = m_2$  or  $m_1 = m_2 = 0$  [57]. The charged-current interactions of  $\nu_e$ 's with electrons in a star can also be taken into account by introducing the additional potential energy  $\sqrt{2}G_F N_e(r) \leq 10^{-12} \text{eV}$ . For the Sun  $N_e(r) = N_0 \exp(-10.54 \frac{r}{R_\odot}) \text{cm}^{-3}$ , where  $N_0$  is the number of electrons at its centre [58]. Assuming  $\Delta m^2 = 0$  for simplicity, the equations of motion become in this case

$$i \left( \frac{\partial}{\partial t} \right) \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \sqrt{2}G_F N_e(r) - \Delta\alpha E\varphi \cos(2\theta) & \frac{\Delta\alpha}{2} E\varphi \sin(2\theta) \\ \frac{\Delta\alpha}{2} E\varphi \sin(2\theta) & 0 \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}.$$

The resonance condition  $\sqrt{2}G_F N_e(r) = E\varphi\Delta\alpha \cos(2\theta)$  is satisfied only when  $\Delta\alpha < 0$  because  $\varphi < 0$ .

## 6. Summary

These lectures have dealt with non-relativistic and relativistic wave equations in weak, external gravitational and inertial fields. Only two fundamental aspects of the interaction have been considered: the generation of quantum phases and spin-gravity coupling.

As shown in Section 2, quantum phases can be calculated exactly to first order in the field and in a manifestly covariant way for Klein-Gordon, Maxwell and Dirac equations. They can then be tested in experiments of increasing accuracy.

The behaviour of quantum systems is consistent with that predicted by general relativity, intended as a theory of both gravity and inertia, down to distances  $\sim 10^{-8} \text{cm}$ . This is borne out of measurements on superconducting electrons ( $\sim 10^{-4} \text{cm}$ ) and on neutrons ( $\sim 10^{-8} \text{cm}$ ) which are not tests of general relativity *per se*, but confirm that the behaviour of inertia and Newtonian gravity is that predicted by wave equations that satisfy the principle of general covariance. Atomic and molecular interferometers will push this limit down to  $10^{-9} - 10^{-11} \text{cm}$  and perhaps lead to new tests of general relativity. Prime candidates are in this regard a correction term to the gravitational red-shift of photons, a (general) relativistic correction to the gravitational field of Earth, and the Lense-Thirring effect of Earth. The

last two experiments may require use of a near space laboratory to obviate the effect of rotation in the first instance and, on the contrary, to sense it in the latter case.

The Mashhoon effect offers interesting insights into the interaction of inertia-gravity with spin. Spin is, of course, a quantum degree of freedom *par excellence*. Spin-rotation coupling plays a fundamental role in precision measurements of the anomalous magnetic moment of muons. These extend the validity of the fully covariant Dirac equation down to distances comparable with the muon's wave-length, or  $\sim 2 \times 10^{-13} \text{cm}$ .

Rotational inertia does not couple universally to matter and does therefore violate the weak equivalence principle. This generates particle helicity oscillations that may play a role in some astrophysical processes.

Other violations of the equivalence principle may play a role in neutrino oscillations. Even small violations,  $\Delta\alpha \sim 10^{-14}$ , could be amplified by a concomitant gravitational field. The resulting oscillations would then become comparable in magnitude with those due to the MSW effect.

More serious violations of the equivalence principle would invalidate the principle of general covariance and render problematic the use of covariant wave equations.

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